$$R(s)z(t,z_0) \approx w[t,R(t)R(s)z_0] \approx w\{t,R(t)\gamma[t,R(s)z_0]\} = z[t,R(s)z_0]$$

which is [22].

The functional equation can be visualized easily, since by [15], $R(t)z$ is equivalent with $S(t)x$, $S(t)y$ for $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $S(t)x$ represents a rotation of the position vector x around the origin through the angle $-t$. If z_0 is not too near P_1 or P_2, then by [22] one can generate and dynamically describe a family of solutions $z[t,R(s)z_0]$ of the restricted three-body problem [5] which begin near P_1 and end near P_2 for $t_1 \leqq t \leqq t_2$ ($t_1 < 0 < t_2$), for instance, using s as parameter. It is not difficult using the foregoing consideration and [21] practically to determine suitable ranges for s, t_1, t_2, if $z(t, z_0)$ is known.

It is finally remarked that [22] is not purely local in nature but describes a property of the trajectories of [2] and thus also of [1] "in the large." This demonstrates at least one use of [17] and [20] and is related to the physical meaning of $\tilde{F}(t,\zeta)$ mentioned before [14]. Further investigation of the properties of this function as given by [12] deserves an effort in view of [9, 11, 13], and of Ref. 1.

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Linearized Steady Motion of Pluri-Reacting Mixtures[1]

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In Ref. 1 it has been proved that, when the chemical characteristic time is much smaller than the macroscopic characteristic time, the linearized steady flow of a singly reacting mixture is equivalent to that of an inert gas mixture with volume viscosity. Parallel statements for unsteady motion in the so-called acoustical approximation are proved in Ref. 2.

It is the purpose of this note to prove two similar statements for the linearized steady motion of mixture in which n reactions take place; namely: 1) when all pertinent relaxation times are much smaller than the macroscopic characteristic time, the linearized steady flow of a pluri-reacting mixture is equivalent to that of an inert gas mixture with a volume viscosity depending on the equilibrium state of the mixture; 2) when m relaxation times ($1 \leq m < n$) are much smaller than the macroscopic characteristic time, the linearized steady flow of an n reacting mixture is equivalent to that of a "viscous" mixture in which ($n - m$) reactions take place, the equivalent kinematic volume viscosity being the same as that for case 1.

To prove these statements, one describes the local thermodynamic state of the mixture in terms of the independent variables v, s, $\mathbf{A}$, where v and s are specific volume and entropy, respectively, and $\mathbf{A}$ is an n dimensional vector whose n components A_i are the affinities of the n reactions taking place in the mixture.

All needed thermodynamic information is then given by the first-order homogeneous function $\varphi = \varphi(\mathbf{A}, s, v)$ (the "fundamental relation"), where $\varphi = e - \mathbf{A}\cdot\boldsymbol{\xi}$ is a thermodynamic potential defined (see Ref. 3) as the nth order Lagrange transform of the specific energy (e) with respect to the n dimensional vector $\boldsymbol{\xi}$ whose ith component ξ_i is the progress variable of the ith reaction. The conjugate dependent quantities $\boldsymbol{\xi}$, T (temperature), and p (pressure) are functions of the basic set ($\mathbf{A}$, s, v) and are defined in terms of the partial derivatives of φ according to the Gibbs relation:

$$d\varphi = -\boldsymbol{\xi}\cdot d\mathbf{A} - pdv + Tds \qquad [1]$$

In the fluid-dynamic evolution of a system not too far from equilibrium conditions, the local thermodynamic state can be considered as a first-order deviation (subscript 1) from an equilibrium state (subscript 0), identified by the set $(0, v_0, s_0)$. The first-order deviation, when all molecular transport effects are negligible, is identified by the set $(\mathbf{A}_1, v_1, 0)$; the first-order disturbances $\boldsymbol{\xi}_1$, p_1, T_1 are linear combinations of $(\mathbf{A}_1, v_1, 0)$ through the second-order derivatives of φ computed at the "point" $(0, v_0, s_0)$. Introducing the $(n \times n)$ matrix $\mathbf{\Phi}$ and the n dimensional vector $\boldsymbol{\varphi}$ defined by

$$\phi_{ij} = -(\partial^2\varphi/\partial A_i\partial A_j)_0 \qquad \varphi_i = (\partial^2\varphi/\partial A_i\partial v) \qquad i, j\colon 1, \ldots n \qquad [2]$$

where the subscript (0) indicates values computed at $(0, v_0, s_0)$, it is

$$p_1 = -\boldsymbol{\varphi}\cdot\mathbf{A}_1 - (a_{e0}{}^2/v_0)(v_1/v_0) \qquad [2a]$$

$$\boldsymbol{\xi}_1 = \mathbf{\Phi}\cdot\mathbf{A}_1 - \boldsymbol{\varphi}v_1 \qquad [2b]$$

where $a_{e0}{}^2 = v_0{}^2(\partial^2\varphi/\partial v^2)_{\mathbf{A}=0}$ is, by definition, the "equilibrium" speed of sound. The matrix $\mathbf{\Phi}$ is positive definite upon the intrinsic stability of the system.

The basic conservation equations for the first-order, steady disturbances introduced by initial and/or boundary conditions into an otherwise constant property equilibrium flow field can then be written, if $\mathbf{V}$ is the velocity vector and $D/Dt = \mathbf{V}_0\cdot\nabla$, as

$$\begin{gathered}(D\mathbf{V}_1/Dt) - (a_{e0}{}^2/v_0)\nabla v_1 = \nabla(\boldsymbol{\varphi}\cdot\mathbf{A}_1) \\ Dv_1/Dt = v_0\nabla\cdot\mathbf{V}_1 \qquad Ds_1/Dt = 0 \qquad [3] \\ D\mathbf{A}_1/Dt = -\mathbf{\Phi}^{-1}\cdot\mathbf{L}\cdot\mathbf{A}_1 + \mathbf{\Phi}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt)\end{gathered}$$

where $\mathbf{\Phi}^{-1}$ is the inverse matrix of $\mathbf{\Phi}$, and the last equation follows from the expression $D\boldsymbol{\xi}_1/Dt = -\mathbf{L}\cdot\mathbf{A}_1$ for the substantial time rate of change of $\boldsymbol{\xi}_1$ and from Eq. [2b]. The $(n \times n)$ matrix $\mathbf{L} = \mathbf{L}(v_0, s_0)$ is positive definite (upon the positive character of the entropy production) and symmetric (upon the Onsager's relation).

Eqs. [1–3] make it possible to prove the forementioned statements. Identify first the "pertinent" relaxation times. Diagonalize the matrix $\mathbf{L}^{-1}\cdot\mathbf{\Phi}$ (where $\mathbf{L}^{-1}$ is the inverse of the matrix $\mathbf{L}$) through a similarity transformation, i.e., let

$$\mathbf{B}_1 = \mathbf{N}\cdot\mathbf{A}_1 \qquad \mathbf{N}\cdot(\mathbf{L}^{-1}\cdot\mathbf{\Phi})\cdot\mathbf{N}^{-1} = \mathbf{\Lambda} \qquad (\Lambda_{ij} = \lambda_i\delta_{ij})$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

Performing these substitutions, the last of Eqs. [3] becomes

$$\mathbf{B}_1 = \mathbf{N}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) - \mathbf{\Lambda}\cdot(D\mathbf{B}_1/Dt) \qquad [4]$$

which shows that $\mathbf{\Lambda}$ has the dimension of time. On the other hand, by known matrix theorems, the λ_i's are the n roots of the determinantal equation

$$|\mathbf{L}^{-1}\cdot\mathbf{\Phi} - \lambda\mathbf{U}| = 0$$

($\mathbf{U}$ is the unit matrix), and, since both $\mathbf{\Phi}$ and $\mathbf{L}^{-1}$ are positive

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definite symmetric matrices, they are all different and positive. Thus the λ_i are the pertinent "relaxation" times of the process. If (l_r/V_r) is a reference macroscopic characteristic time, then the conditions of statement 1 amount to the inequalities $(\lambda_i l_r/V_r) \ll 1$ for any i between 1 and n. Consequently, the time rate of change of $\mathbf{B}_1$ can be neglected with respect to $\mathbf{B}_1$ in Eq. [4], which, accounting the second of Eqs. [3], will accordingly yield

$$\mathbf{A}_1 = \mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) = v_0(\mathbf{L}^{-1}\cdot\boldsymbol{\varphi})\nabla\cdot\mathbf{V}_1$$

Substituting this value of $\mathbf{A}_1$ into Eq. [2a] results in

$$p_1 = -(\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi})v_0\nabla\cdot\mathbf{V}_1 - (a_{e0}^2/v_0^2)v_1$$

The positive definite character of the matrix $\mathbf{L}$ implies that $(\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}) > 0$. One can then define an essentially positive kinematic volume viscosity ν_v as

$$\nu_v = v_0(\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}) \qquad [5]$$

and the basic system becomes, simply

$$\begin{aligned} Dv_1/Dt &= v_0\nabla\cdot\mathbf{V}_1 \\ D\mathbf{V}_1/Dt &= v_0\nabla[\nu_v\nabla\cdot\mathbf{V}_1 + (a_{e0}^2/v_0^2)v_1] \end{aligned}$$

thus proving statement 1.

In the conditions of statement 2, it is $(\lambda_i l_r/V_r) \ll 1$ for $1 \leq i \leq m$ and $(\lambda_i l_r/V_r) = O(1)$ for $m + 1 \leq i \leq n$, and, accordingly, the contribution of the first m elements in the diagonal matrix $\mathbf{B}_1$ is negligible.

If the n dimensional vector $\mathbf{B}_1$ is considered as sum of an m dimensional vector $\mathbf{F}$ and an $(n-m)$ dimensional vector $\mathbf{G}$ defined by

$$F_\alpha = B_{1\alpha} \qquad G_\gamma = B_{1\gamma} \qquad \begin{array}{r} 1 \leq \alpha \leq m \\ m+1 \leq \gamma \leq n \end{array}$$

and if one introduces the following partitions of the matrices $\mathbf{N}$ and $\mathbf{N}^{-1}$:

$$\begin{array}{llr} M_{\alpha i} = N_{\alpha i} & Z_{\gamma i} = N_{\gamma i} & 1 \leq i \leq n \\ & & 1 \leq \alpha \leq m \\ R_{i\alpha} = N_{i\alpha}^{-1} & \Omega_{i\gamma} = N_{i\gamma}^{-1} & m+1 \leq \gamma \leq n \end{array}$$

then Eq. [4] can be split in the following two equations:

$$\begin{aligned} \mathbf{F} &= \mathbf{M}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) \\ \mathbf{G} &= \mathbf{Z}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) - \boldsymbol{\Lambda}\cdot(D\mathbf{G}/Dt) \end{aligned}$$

and the expression for p_1 becomes, successively, through Eq. [5]

$$\begin{aligned} p_1 + (a_{e0}^2/v_0^2)v_1 &= -\boldsymbol{\varphi}\cdot\mathbf{N}^{-1}\cdot\mathbf{B}_1 = -\boldsymbol{\varphi}\cdot[\mathbf{R}\cdot\mathbf{F} + \boldsymbol{\Omega}\cdot\mathbf{G}] \\ &= -\boldsymbol{\varphi}\cdot\mathbf{L}^{-1}\cdot\boldsymbol{\varphi}(Dv_1/Dt) + \boldsymbol{\varphi}\cdot\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\cdot(D\mathbf{G}/Dt) \\ &= -\nu_v\nabla\cdot\mathbf{V}_1 + \boldsymbol{\varphi}\cdot\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}\cdot(D\mathbf{G}/Dt) \end{aligned}$$

thus proving (statement 2) that the subject motion is equivalent to the motion of a viscous mixture in which only $(n - m)$ reactions take place. The basic unknowns associated with the reactions are, in general, the $(n - m)$ linear combination of the affinities A_i given by

$$G_\gamma = \sum_{i=1}^{n} N_{\gamma i}^{-1}A_i \qquad m + 1 \leq \gamma \leq n$$

wherein some of the components $N_{\gamma i}^{-1}$ may turn out to be negligible, for they are functions of the λ_i's, and, in the present case, it is $(\lambda_\alpha/\lambda_\gamma) \ll 1$. In particular, if the reactions are uncoupled (i.e., $L_{ij} = 0$ for $i \neq j$), the G_γ's reduce to the $(n - m)$ affinities A_γ's.

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Stagnation Region in Rarefied, High Mach Number Flow[1]

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The departure at low Reynolds number from the concept of a thin boundary layer flow has been studied in Refs. 1–6 and in other works. In Refs. 2–5, the linear viscosity-temperature relation greatly simplifies the analyses, but the resultant stagnation point heat transfer rates obtained for the vorticity-interaction regime appear to differ considerably from that of Hayes and Probstein (1),[4] who adopt a Sutherland viscosity law. This difference suggests that the linear viscosity temperature relation may not be adequate for low Reynolds number studies. However, in this note it will be demonstrated that the linear representation of the viscosity law, with an appropriate reference temperature, is adequate for most purposes in both the high and the low Reynolds number regimes.[5] Also included is a comparison of the results of viscous shock-layer theory (5,6) with the recent heat transfer measurements of Ferri and Zakkay (7).

Viscosity-Temperature Relation

In Ref. 5, the viscosity-temperature relation is represented by the linear law

$$\mu = \mu_*(T/T_*) \qquad [1]$$

where μ_* is the viscosity evaluated at the reference temperature T_*. This latter quantity is chosen as

$$T_* = (T_s + T_w)/2 \qquad [2]$$

for the stagnation region, where T_s and T_w are the temperatures immediately behind the shock and at the body surface, respectively.

Simplifications of this type may not always lead to accurate solutions, especially for the details of the flow field near a cold surface. However, Eckert (8) and others have shown that this linear representation is generally adequate for predicting the skin friction and heat transfer characteristics of compressible boundary layers if the reference temperature is chosen at an appropriate level. When specialized to the stagnation point boundary layer, Eckert's reference temperature is $T_* = (T_0 + T_w)/2$, where T_0 is the stagnation temperature. The reference temperature is essentially that given by Eq. [2], since for this case $T_s \approx T_0$.

In the following discussion, the particular case of $\mu \propto T^\omega$ is compared with the case of the linear viscosity-temperature law, Eq. [1]. The exponent ω is taken to be 0.65, which compares reasonably well with the Sutherland law in the range of $T = 500°$ to 3000°K for air. Numerical solutions of the viscous shock-layer equations (5) for the axisymmetric stagnation region (based on a model of ideal gas with constant specific heats) have been obtained using the nonlinear viscosity law for $Pr = 0.71$, $\epsilon = p/2\rho h = (\gamma - 1)/2\gamma = 0.10$ and $T_w/T_0 = 0.10$ over a wide range of the Reynolds number

$$Re_b = \rho_\infty U_\infty a/\mu_0 \qquad [3]$$

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[4] Numbers in parentheses indicate References at end of paper.

[5] The reason for the difference between the results of Ref. 1 and others is discussed in Ref. 6.